MULTIPLIER SPECTRA AND THE MODULI SPACE OF DEGREE 3 MORPHISMS ON $\mathbb{P}^1$

BENJAMIN HUTZ AND MICHAEL TEPPER

Abstract. The moduli space of degree d morphisms on $\mathbb{P}^1$ has received much study. McMullen showed that, except for certain families of Lattès maps, there is a finite-to-one correspondence (over $\mathbb{C}$) between classes of morphisms in the moduli space and the multipliers of the periodic points. For degree 2 morphisms Milnor (over $\mathbb{C}$) and Silverman (over $\mathbb{Z}$) showed that the correspondence is an isomorphism [7, 8]. In this article we address two cases: polynomial maps of any degree and rational maps of degree 3.

1. Introduction

Let Hom_d be the space of degree d endomorphisms of $\mathbb{P}^1$. Let $\phi \in \mathrm{Hom}_d$, then after choosing coordinates for $\mathbb{P}^1$ we may represent the coordinates of ϕ as two degree d homogeneous polynomials with no common zeros. We may consider $\mathrm{Hom}_d \subset \mathbb{P}^{2d+1}$ by identifying a morphism ϕ with its set of coefficients. There is a natural action of PGL_2 by conjugation on Hom_d, which extends to $\mathbb{P}^{2d+1}$, and we get a moduli space $M_d = \mathrm{Hom}_d / \mathrm{PGL}_2$ [4, 8]. The moduli space M_d, and its generalization to $\mathbb{P}^N$, has received much study, see for instance [2, 4, 5, 6, 7, 8]. Milnor [7] gave an isomorphism $M_2 \cong \mathbb{A}^2$ over $\mathbb{C}$ which Silverman [8] extended to $\mathbb{Z}$ in addition to showing that M_d is an affine integral scheme over $\mathbb{Z}$. However, for $d > 2$ less is known about the structure of M_d. McMullen [6] showed that there is a finite-to-one correspondence (over $\mathbb{C}$) between classes of morphisms in the moduli space and certain conjugation invariants called multipliers. It is this correspondence of McMullen that we study in this article. We next supply the necessary definitions to state the correspondence precisely.

Definition 1. Let $\phi \in \mathrm{Hom}_d$ and $\mathrm{Per}_n(\phi) = \{P \in \mathbb{P}^1 : \phi^n(P) = P\}$ be the set of periodic points of period n for ϕ. For $P \in \mathrm{Per}_n(\phi)$, ϕ^n induces a map on the cotangent space of $\mathbb{P}^1$ at P to itself. The cotangent space is dimension 1 in this case, so the induced map is an element of GL_1 (a scalar) and we call it the *multiplier* at P and is denoted $\lambda_P(\phi)$.

Define the *n-multiplier spectrum* $\Lambda_n = \{\lambda_P(\phi) : P \in \mathrm{Per}_n(\phi)\}$, where the multipliers are taken with appropriate multiplicity.

Define $\sigma_{n,i}$ for $1 \leq i \leq d^n + 1$ as the i^{th} symmetric function on the n-multiplier spectrum. We denote the $(d^n + 1)$-tuple $\boldsymbol{\sigma}_n = (\sigma_{n,1}, \dots, \sigma_{n,d^n+1})$.

It is an easy chain rule exercise to show that, as an unordered set, Λ_n is invariant under conjugation. The multipliers depend algebraically, but not rationally, on the coefficients on ϕ, however, the $\sigma_{n,i}$ are actually rational functions in the coefficients of ϕ [7, 8]. Furthermore, the $\sigma_{n,i}$ are in fact regular functions on M_d [8].

Definition 2. Define the map

$$\begin{aligned} \tau_{d,n} : M_d &\to \mathbb{A}^k \\ [\phi] &\mapsto (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \dots, \boldsymbol{\sigma}_n). \end{aligned}$$

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We define the *degree of* $\tau_{d,n}$ as the number of points in $\tau_{d,n}^{-1}(P)$ for a generic point P in $\tau_{d,n}(M_d)$. From [9, Theorem 4.54] the degree of $\tau_{d,n}$ stabilizes as $n \to \infty$ and we write $\deg(\tau_d)$ for this value.

Specifically, McMullen [6, Corollary 2.3] showed that the conjugacy class of $\phi \in \mathrm{Hom}_d$ is determined up to finitely many choices by its multiplier spectra if ϕ is not a flexible Lattès map. This is typically stated as the following theorem.

Theorem 3. [6, Corollary 2.3] *Fix $d \geq 2$. For n sufficiently large, $\tau_{d,n}$ is finite-to-one on $M_d(\mathbb{C})$ except for certain families of Lattès maps.*

Two maps which have the same set of multipliers are called *isospectral*. All flexible Lattès maps (integer multiplication on elliptic curves) are isospectral since they all have the same multiplier spectra. In other words, the regular maps $\sigma_{n,i}$ are constant on the family of flexible Lattès maps. Additionally, one can use rigid Lattès maps to show that $\deg(\tau_d) \geq C_\epsilon d^{\frac{1}{2}-\epsilon}$ for some constant C. In particular, for squarefree d, $\deg(\tau_d) \geq h$ where h is the class number of $\mathbb{Q}(\sqrt{-d})$ [9, §6.6]. In this article we address two cases on $\mathbb{P}^1$: polynomial maps of any degree and rational maps of degree 3.

2. Results

In Section 3 we examine the locus of polynomial maps in M_d. A *polynomial map* is a map with a totally ramified fixed point. We denote $\tilde{\tau}_{d,n}$ as the restriction of $\tau_{d,n}$ to the space of polynomial maps of degree d. We show

Theorem 4. *Except for possibly finitely many fibers, the map $\tilde{\tau}_{d,1}$ is finite-to-one.*

For $d = 2$, 3, 4, and 5 we compute that there are no exceptional fibers and for $d = 2$ and 3 the correspondence is 1-to-1, for $d = 4$ the correspondence is 2-to-1, and for $d = 5$ the correspondence is 6-to-1. Furthermore, we show that $\tilde{\tau}_{d,n}$ is one-to-one for $d = 2$, 3, 4, and 5, when considering the 2-multiplier spectra. That is $\deg(\tilde{\tau}_d) = 1$ for $d = 2$, 3, 4, and 5.

Theorem 5. *The maps $\tilde{\tau}_{4,2}$ and $\tilde{\tau}_{5,2}$ are one-to-one.*

In Section 4 we consider rational maps and examine M_3. Our main result is computing the degree of the correspondence $\tau_{3,2}$ (the number of isospectral maps up to conjugation equivalence) using groebner bases.

Theorem 6. *The value* $\deg(\tau_{3,2}) = 12$.

While this value represents an upper bound on $\deg(\tau_3)$ it is not necessarily equal to $\deg(\tau_3)$. See Example 14 for an example where additional multiplier information cases the degree to decrease. While our methods theoretically would allow us to compute $\deg(\tau_3)$, the computations with additional multiplier information did not finish.

An interesting open question is to determine what information uniquely specifies a class in M_d. Towards this end, in Section 5 we restrict to the case of a generic map, where generic is the open dense subset of M_d of maps with distinct fixed points.

Theorem 7. *If ϕ has distinct fixed points, then the following correspondence is one-to-one:*

$$\tau_{d,1}^{+} : M_d \to \mathbb{A}^{2d+1}$$
$$[\phi] \mapsto (\boldsymbol{\sigma}_1, \mathrm{Per}_1(\phi)).$$

3. Polynomials

We first examine polynomial maps, ϕ with a totally ramified fixed point.

Definition 8. We denote $P_d \subset M_d$ as the moduli space of degree d polynomial maps.

Define the restriction of $\tau_{d,n}$ to the space of polynomial maps as

$$\tilde{\tau}_{d,n} : P_d \to \mathbb{A}^k.$$

Any given polynomial map may be conjugated to the form

$$z^d + a_2 z^{d-2} + \cdots + a_d. \tag{1}$$

Note however that this is not a normal form since two such polynomials may be conjugate. We are interested in the correspondence coming from McMullen's theorem

$$\begin{aligned}\tilde{\tau}_{d,n} &: P_d \to \mathbb{A}^k \\ &[\phi] \mapsto (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n).\end{aligned}$$

3.1. **Multipliers of fixed points.** We first extend the relation $\sigma_{1,1} = \sigma_{1,3} + 2$ in M_2 to M_d.

Theorem 9. *Let $[\phi] \in M_d$. The symmetric functions $\boldsymbol{\sigma}_1$ satisfy*

$$(-1)^{d+2}\sigma_{1,d+1} + (-1)^{d-1}\sigma_{1,d-1} + (-1)^{d-2}2\sigma_{1,d-2} + \cdots - (d-1)\sigma_{1,1} + d = 0.$$

Proof. Label the multipliers of the fixed points as $\{\lambda_0, \ldots, \lambda_d\}$. Assume first that the fixed points are distinct. Distinct fixed points implies the multipliers are all different from 1 and we may apply the relation [9, Theorem 1.14]

$$\sum_{i=0}^{d} \frac{1}{1-\lambda_i} = 1.$$

Clearing denominators we have

$$\sum_{j=0}^{d} \prod_{i \neq j} (1-\lambda_i) = \prod_{i=0}^{d} (1-\lambda_i).$$

Expanding the right-hand side we have

$$\sum_{j=0}^{d} \prod_{i \neq j} (1-\lambda_i) = 1 - \sigma_{1,1} + \sigma_{1,2} + \cdots + (-1)^d \sigma_{1,d} + (-1)^{d+1}\sigma_{1,d+1}.$$

Now we expand the left-hand side. The term of degree $n \geq 0$ in the λ_i's is $(-1)^n(d+1-n)\sigma_{1,n}$, since each term of $\sigma_{1,i}$ is missing exactly from n terms of the sum, for notational convenience we define $\sigma_{1,0} = 1$. We rewrite this as

$$\sum_{i=0}^{d} (-1)^i (d+1-i)\sigma_{1,i} = \sum_{i=0}^{d+1} (-1)^i \sigma_{1,i}$$

Combining with the right-hand side we have the desired result when ϕ has distinct fixed points.

The set of ϕ with $1 \in \Lambda_1$ is a Zariski closed set, so the ϕ with distinct fixed points are dense in Hom_d. Thus, the function

$$\sum_{i=0}^{d} (-1)^i (d+1-i)\sigma_{1,i} - \sum_{i=0}^{d+1} (-1)^i \sigma_{1,i}$$

is identically zero. $\square$

Corollary 10. *For $[\phi] \in P_d$ we have*

$$(-1)^{d-1}\sigma_{1,d-1} + (-1)^{d-2}2\sigma_{1,d-2} + \cdots - (d-1)\sigma_{1,1} + d = 0.$$

Proof. The fixed point at ∞ has $\lambda = 0$ and hence $\sigma_{1,d+1} = \prod_i \lambda_i = 0$. $\square$

We wish to show that specifying $\boldsymbol{\sigma}_1 \in \tilde{\tau}_{d,1}(\mathbb{A}^{d+1})$ determines a polynomial map in $P_d \subset M_d$ up to finitely many choices.

Theorem 11. *Let $\phi \in P_d$ with affine fixed points $\{z_1, \ldots, z_d\}$. Each $\lambda_i \in \Lambda_1$ such that $\lambda_i \neq 1$ determines a homogeneous equation of degree $d-1$ of the form*

$$F_i(z_1, \ldots, z_d) = \lambda_i - 1.$$

Proof. For a polynomial map $\phi(z)$ we may write

$$\phi(z) - z = \prod_{i=1}^{d} (z - z_i).$$

If $\lambda_i = 1$, then z_i is a multiple root of the above equation and upon taking derivatives we get a totality. For $\lambda_i \neq 1$ we compute

$$\phi'(z_i) - 1 = \lambda_i - 1 = \prod_{j=1, j\neq i}^{d} (z_i - z_j).$$

Thus, we get

$$F_i(z_1, \ldots, z_d) = \lambda_i - 1$$

for a homogeneous equation $F_i(z_1, \ldots, z_d)$ of degree $d-1$. □

Theorem 12. *Except for possibly finitely many fibers, the map*

$$\tilde{\tau}_{d,1} : P_d \to \mathbb{A}^{d+1}$$
$$[\phi] \mapsto \boldsymbol{\sigma}_1$$

is finite-to-one.

Proof. We take as our starting point the system of homogeneous equations of degree $d-1$

$$F_i(z_1, \ldots, z_d) = \lambda_i - 1$$

from Theorem 11. Label the hypersurfaces as $H_i = V(F_i - \lambda_i + 1)$. We proceed one λ_i at a time. Except for finitely many λ_2, the hypersurfaces H_1 and H_2 intersect properly (codimension 2) since H_1 has finitely many components. Similarly, except for finitely many λ_3 the varieties $(H_1 \cap H_2)$ and H_3 intersect properly (co-dimension 3). We proceed similarly for the remaining H_i avoiding at most finitely many λ_i. Thus, we have a codimension d set in a dimension d space, so there are only finitely many solutions. The solutions are the possible sets of fixed points of the polynomial map ϕ, and the set of fixed points determines a unique polynomial map. □

We compute the degree of the correspondence $\tilde{\tau}_{d,1}$ for $d = 2$, 3, 4, and 5 using groebner bases for the system of equations from Theorem 11 for a polynomial with distinct fixed points.

Theorem 13. *The value*

$$\deg(\tilde{\tau}_{d,1}) = \begin{cases} 1 & d = 2 \\ 1 & d = 3 \\ 2 & d = 4 \\ 6 & d = 5. \end{cases}$$

Proof. The case $d = 2$ was done by Milnor [7]. The cases $d = 3$, 4, and 5 are groebner basis calculations and were performed in Singular [3]. For $d = 3$ and 4 we are able to compute the groebner basis of the system of equations from Theorem 11 with the λ_i as indeterminants. This produces $(d-1)!$ configurations of fixed points. However, these are not all distinct up to conjugation because if $\{z_i\}$ is a set of fixed points, then so is $\{\zeta_{d-1} z_i\}$ where ζ_{d-1} is a primitive $(d-1)^{\text{st}}$ root of

unity. This is readily apparent from the form of the groebner basis when using the lexicographic ordering for elimination. Thus, for $d = 3$ we have $(3-1)!/2 = 1$ and for $d = 4$ we have $(4-1)!/3 = 2$ distinct conjugacy classes in the moduli space.

For $d = 5$, the groebner basis calculation with the λ_i as indeterminants did not finish, so we employed an alterative method. The method is to pick a specialization of the λ_i and compute the degree, and then show that this is in fact a generic value by showing that it is constant under perturbation of the λ_i. Singular was able to compute the groebner basis fixing any three of the four λ_i for $\Lambda_1 = \{-2, -3, -4, 8, 0\}$. In all four cases, there were 24 solutions of fixed point arrangements some of which differ by a 4^{th} root of unity. Thus, there are $24/4 = 6$ distinct conjugacy classes in the moduli space. □

Remark. For $d = 6$ and fixed Λ_1, the computation would finish modulo primes, but not in any general situation. It appears that there are at least 1900 solutions of fixed point arrangements for $d = 6$, so we expect a large jump in degree from $d = 5$ to $d = 6$.

Theorem 13 says that Λ_1 specifies a polynomial up to finitely many choices. The next example shows that Λ_2 can further distinguish between polynomials.

Example 14. Consider the map $\tilde{\tau}_{4,1}(P_4) \to \mathbb{A}^5$ and the fiber

$$\tilde{\tau}_{4,1}^{-1}(-1724, -1163982, 74470803, 4530821869, 0).$$

In other words $\Lambda_1 = \{-2243, -59, 0, 67, 511\}$. There are two polynomials (up to conjugation) in this inverse image

$$\begin{aligned} f(z) &= z^4 - 77z^2 + 217z - 140 \\ g(z) &= z^4 - 721/8z^2 + 217z + 165025/256. \end{aligned}$$

However,

$$\tilde{\tau}_{4,2}(f) \neq \tilde{\tau}_{4,2}(g).$$

Theorem 15. $\deg(\tilde{\tau}_4) = \deg(\tilde{\tau}_{4,2}) = 1$.

Proof. We use a groebner basis calculation in Magma [1]. To the fixed point equations, we add equations for a single 2-periodic point

$$\phi(\phi(\beta)) = \beta \qquad \text{and} \qquad (\phi^2)'(\beta) = \lambda_\beta.$$

From just the fixed point equations there were 6 distinct choices of the set of fixed points (only 2 up to conjugation). For each set of fixed points there are 16 possible 2-periodic points, and, hence, there are at least 96 points on this variety. Specializing to $\Lambda_1 = \{-5, 5, 4, 7/5, 0\}$ and using Magma we have a zero-dimensional scheme in the coordinates (β, λ_β) which is reduced with 96 distinct points. Working modulo 13 (where the scheme is still reduced) we determine the 96 points over $\mathbb{F}_{13^{60}}$ and that there are in fact 2 distinct possibilities for Λ_2. Thus, there is one polynomial map associated to $\{\Lambda_1, \Lambda_2\}$. Since these points are all multiplicity one, there will remain 2 distinct maps under perturbation of Λ_1.

□

Theorem 16. $\deg(\tilde{\tau}_5) = \deg(\tilde{\tau}_{5,2}) = 1$.

Proof. We again use a groebner basis calculation in Magma [1]. To the fixed point equations, we add equations for a single 2-periodic point

$$\phi(\phi(\beta)) = \beta \qquad \text{and} \qquad (\phi^2)'(\beta) = \lambda_\beta.$$

From just the fixed point equations there were 24 distinct choices of the set of fixed points (6 up to conjugation). For each set of fixed points there are 25 possible 2-periodic points, and, hence,

there are at least 600 points on this variety. Specializing to $\Lambda_1 = \{-5, 5, -4, -2, 29/9, 0\}$ and using Magma we have a zero-dimensional scheme in the coordinates (β, λ_β). Working modulo 29 the scheme is reduced with 600 distinct points over $\mathbb{F}_{29^{240}}$. There are in fact 6 distinct possibilities for Λ_2. Thus, there is one polynomial map associated to $\{\Lambda_1, \Lambda_2\}$. Since these points are all multiplicity one, there will remain 6 distinct maps under perturbation of Λ_1.

□

Remark. It would be interesting to determine if $\tilde{\tau}_{d,2}$ (or $\tilde{\tau}_{d,n}$) is generically one-to-one on polynomials or if the degree will eventually grow.

3.2. **Explicit P_3^1.** Milnor [7] gave an explicit normal form for classes in M_2 in terms of Λ_1. We give a similar description for P_3. In particular, we give an explicit description of the fiber $\tilde{\tau}_{3,1}^{-1}(\boldsymbol{\sigma}_1) \subset P_3$.

Theorem 17. *Let $[\phi] \in \tilde{\tau}_{d,1}^{-1}(\boldsymbol{\sigma}_1)$ and write $[\phi]$ in the form $\phi(z) = z^3 + az + b$. If ϕ has 3 distinct affine fixed points with multipliers λ_1, λ_2, and λ_3. Then,*

$$\begin{aligned} a &= -\frac{(\lambda_1^2 + (\lambda_2 - 6)\lambda_1 + (\lambda_2^2 - 6\lambda_2 + 9))}{(3\lambda_1 + (3\lambda_2 - 6))} \\ 27b^2 &= \sigma_{1,3} - \sigma_{1,2}a + \sigma_{1,1}a^2 - a^3. \end{aligned}$$

If ϕ has 2 distinct affine fixed points with λ the multiplier that is not 1. Then,

$$\begin{aligned} a &= 1 - \frac{\lambda - 1}{3} \\ b &= -2\left(\pm\frac{\lambda - 1}{9}\right)^{3/2}. \end{aligned}$$

If ϕ has a single affine fixed point, then

$$a = b = 0.$$

Proof. Specifying $\boldsymbol{\sigma}_1$ in fact specifies the 1-multiplier spectrum $\Lambda_1 = \{\lambda_1, \lambda_2, \lambda_3, 0\}$.

Case 1 (3 distinct affine fixed points). Then we know for the three affine fixed points that

$$z_i = \pm\sqrt{\frac{\lambda_i - a}{3}}.$$

Additionally, we have

$$z^3 + (a-1)z + b = \prod_{i=1}^{3}(z - z_i)$$

and we have

$$\begin{aligned} b &= -\prod_{i=1}^{3} z_i = -\prod_{i=1}^{3} \pm\sqrt{\frac{\lambda_i - a}{3}} \\ (a-1) &= z_1z_2 + z_1z_3 + z_2z_3 \\ 0 &= z_1 + z_2 + z_3 = \sum_{i=1}^{3} \pm\sqrt{\frac{\lambda_i - a}{3}}. \end{aligned}$$

Solving, we arrive at

$$\begin{aligned} a &= -\frac{(\lambda_1^2 + (\lambda_2 - 6)\lambda_1 + (\lambda_2^2 - 6\lambda_2 + 9))}{(3\lambda_1 + (3\lambda_2 - 6))} \\ 27b^2 &= \sigma_3 - \sigma_2 a + \sigma_1 a^2 - a^3. \end{aligned}$$

Case 2 (2 distinct affine fixed points). Let $\lambda_3 = \lambda$ be the multiplier that is not equal to 1, then since $z_1 + z_2 + z_3 = 0$ and $z_1 = z_2$ we know that

$$z_3 = -2z_1$$

and that

$$\phi(z) - z = z^3 + (a-1)z + b = (z - z_1)^2(z + 2z_1).$$

Thus, we have

$$\begin{aligned} a &= 1 - 3z_1^2 \\ b &= 2z_1^3 \\ \lambda &= 3z_3^2 + a = 12z_1^2 + a = 9z_1^2 + 1. \end{aligned}$$

Thus, we solve

$$z_1 = \pm\sqrt{\frac{\lambda - 1}{9}}$$

and

$$\begin{aligned} a &= 1 - \frac{\lambda - 1}{3} \\ b &= \pm 2\left(\frac{\lambda - 1}{9}\right)^{3/2}. \end{aligned}$$

Case 3 (1 distinct affine fixed point). If $\lambda_i = 1$ for $i = 1, 2, 3$, then we must have $z_1 = z_2 = z_3$ for the three affine fixed points since they must all have multiplicity at least 2. Since also $z_1 + z_2 + z_3 = 0$, we must have $z_i = 0$ and, hence, $\phi(z) = z^3$.

□

Similar to the computation that $\phi_c(z) = z^2 + c$ is the family $\boldsymbol{\sigma}_1 = (2, 4c, 0)$ in M_2^1 we can compute the image of degree 3 polynomials.

Corollary 18. *We may write* $[\phi] \in P_3$ *as*

$$\phi_{a,b}(z) = z^3 + az + b.$$

up to the sign of b*. In particular,*

$$[\phi_{a,b}] = [\phi_{a,-b}].$$

Theorem 19. *The image of* $\phi_{a,b}$ *under* $\tau_{3,1}$ *is given by*

$$\tau_{3,1}(\phi_{a,b}) = \boldsymbol{\sigma}_1 = (6 - 3a, 9 - 6a, 9a - 12a^2 + 4a^3 + 27b^2, 0)$$

Proof. Direct computation (performed with Mathematica [11]). □

4. Rational Maps

Theorem 20. *The value* $\deg(\tau_{3,2}) = 12$.

Proof. The proof proceeds in two steps, first we determine a zero-dimensional variety whose points give the coefficients of the map. Then we determine the number of points on this variety.

Recall that $\deg(\tau_3) = \#(\tau_3^{-1}(P))$ for a generic point P. Generically, there are 4 distinct fixed points ($\lambda_i = 1$ is a closed condition) and at least one 2-periodic point which is not also a fixed point ($\lambda_i \neq -1$).

We dehomogenize and write ϕ as a rational map denoted as $\overline{\phi}(z)$ We label the coefficients of the general such map as

$$\overline{\phi}(z) = \frac{a_1z^3 + a_2z^2 + a_3z + a_4}{b_1z^3 + b_2z^2 + b_3z + b_4}.$$

Under a PGL_2 transformation we may move two of the fixed points to 0 and ∞ to have

$$\overline{\phi}(z) = \frac{a_1 z^3 + a_2 z^2 + a_3 z}{b_2 z^2 + b_3 z + b_4}.$$

Computing the multipliers we also have

$$\lambda_0 b_4 = a_3 \qquad \lambda_\infty a_1 = b_2.$$

So we have left to determine the coefficients $\{a_1, a_2, b_3, b_4\}$. A PGL_2 transformation allows us to move a third fixed point to 1. Then we have

$$a_2 = (b_2 + b_3 + b_4) - a_1 - a_3.$$

Looking at the multiplier at $z = 1$ we can solve (since $\lambda_1 \neq 1$) for

$$b_3 = \frac{(1 - \lambda_1\lambda_\infty)a_1 + (2 - \lambda_0 - \lambda_1)b_4}{\lambda_1 - 1}.$$

Letting α be the fourth (and last) fixed point. Then we can solve for

$$b_4 = \frac{(\alpha a_1 \lambda_\infty - \alpha a_1)}{1 - \lambda_0}.$$

This has provided all of the coefficients of $\overline{\phi}(z)$ in terms of $\{\lambda_0, \lambda_1, \lambda_\infty, \alpha\}$ except for a_1. To account for a_1 we may either consider $\phi(z)$ as a map on $\mathbb{P}^1$ and set $a_1 = 1$ or, equivalently, notice that a_1 is a factor of each of the coefficients and cancels in $\overline{\phi}(z)$. Also, note that λ_α is uniquely determined by the relation [9, Theorem 1.14]

$$\frac{1}{1-\lambda_1} + \frac{1}{1-\lambda_0} + \frac{1}{1-\lambda_\infty} + \frac{1}{1-\lambda_\alpha} = 1.$$

Let $\beta \notin \{0, 1, \infty, \alpha\}$ be an exact 2-periodic point. We have new equations

$$\begin{aligned} \overline{\phi}^2(\beta) &= \beta \\ \lambda_\beta = (\overline{\phi}^2)'(\beta) &= \overline{\phi}'(\beta)\overline{\phi}'(\overline{\phi}(\beta)). \end{aligned} \tag{2}$$

Given λ_β, the system (2) is 2 variables $\{\alpha, \beta\}$ and 2 equations. Except for possibly finitely many choices of λ_β, this system defines a 0-dimensional variety in coordinates (α, β). Thus, for a generic point P, the fiber $\tau_{3,2}^{-1}(P)$ is finite-to-one.

We now determine $\deg(\tau_{3,2})$. Computations were done in Magma [1]. A generic groebner basis computation does not finish. Choosing a particular specialization (values of the multipliers) and computing the degree of the reduced subscheme we find that it has 18 distinct points (in $\mathbb{P}^2$). We will show there remains 18 distinct points under perturbation of the multipliers.

We find 6 points where the map is not generic, $\alpha = 0$ or 1 is a fixed point with multiplicity greater than one. In coordinates (α, β, z), where z is the homogenizing variable, the 6 points are

$$\begin{aligned} P_1 &= (1, 0, 0) \\ P_2 &= (0, 0, 1) \\ P_3 &= (1, 1, 1) \\ P_4 &= \left(0, \frac{-\lambda_1 - \lambda_\infty + 2}{1 - \lambda_1}, 1\right) \\ P_5 &= \left(1, \frac{\lambda_\infty - 1}{1 - \lambda_0}, 1\right) \\ P_6 &= \left(1, -\frac{\lambda_0\lambda_1\lambda_\infty - \lambda_0\lambda_1 - \lambda_\infty + 1}{\lambda_0\lambda_1 - \lambda_0 - \lambda_1 + 1}, 0\right). \end{aligned}$$

By computing the determinant of the Jacobian matrix for P_4, P_5, and P_6 we see that they have generic multiplicity at least 2. We next compute that P_1, P_2, and P_3 each have generic multiplicity 42. The method is to pick a specialization where this occurs, and then show that the multiplicities remain constant under perturbation of the multipliers. Magma is able to compute the multiplicities with 3 of the 4 multipliers fixed, demonstrating that the generic multiplicities are each 42. Since the total number of points of intersection (with multiplicities) by Bézout's theorem is 144, we see that the other twelve points must have multiplicity 1 (and P_4, P_5, and P_6 have multiplicity exactly 2). Thus, these twelve points also remain distinct under perturbation of the multipliers.

The points $P_1, \ldots, P_6$ do satisfy the necessary equations, but they correspond to non-generic maps, $\alpha = 0$ or 1, causing the fixed points to not be distinct. It is easy to check that these six points are all of the possibilities for $\alpha, \beta \in \{0, 1\}$. Similarly, if (α, β) and (α, β') corresponded to the same map, then we would have $\lambda_\beta = \lambda_{\beta'}$ which is a non-generic situation. Thus, the remaining twelve points are inverse images under $\tau_{3,2}$ and the degree of $\tau_{3,2} = 12$. $\square$

However, $\tau_{3,2}$ is not necessarily finite-to-one on fibers with higher multiplicity fixed points.

Example 21. Consider a map $\phi \in M_3$ with two multiplicity 2 fixed points. Move them to 0 and ∞ to get

$$\phi(z) = \frac{a_1 z^3 + a_2 z^2 + a_3 z}{b_1 z^2 + b_3 z + b_3}.$$

Since they are multiplicity 2 then we know $\lambda_1 = \lambda_\infty = 1$.

Now, consider $\phi(\phi(z)) - z$. Solving

$$\begin{aligned} z_1 &= -a_1(a_2/3 + b_3/6) \\ z_1 &= \phi(\phi(z_1)) \end{aligned}$$

we can produce z_1 the only exact 2-periodic point and move it to 1 with a PGL_2 transformation. Since it has multiplicity greater than 1, its multiplier is 1. So we have two equations in the 4 (up to scaling) remaining coefficients $\{a_1, a_2, a_3, b_3\}$ which is not enough to determine ϕ up to finitely many choices.

Remark. The set $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ should be enough for finite-to-one for all $\phi \in M_3$ and is necessary in the case of a map with 1 fixed point, 1 exact 2 periodic point, and 1 exact 3 periodic point.

5. Multipliers and Fixed Points

McMullen's theorem tells us that there are only finitely many maps with a given set of multiplier spectra. An interesting question would be to determine what information is necessary to specify a map uniquely in M_d. In the case of distinct fixed points, the following theorem shows that specifying the fixed points is enough.

Theorem 22. *If ϕ has distinct fixed points, then the following correspondence is 1-to-1:*

$$\begin{aligned} \tau_{d,1}^{+} &: \mathrm{Hom}_d \to \mathbb{A}^{2d+1} \\ [\phi] &\mapsto (\boldsymbol{\sigma}_1, \mathrm{Per}_1(\phi)). \end{aligned}$$

Proof. We write

$$\phi(z) = z - \frac{p(z)}{q(z)}.$$

The fixed points are the roots of the polynomial $p(z)$ so it can be specified (up to constant).

We have

$$\phi'(z) = 1 - \frac{p'(z)}{q(z)} - \frac{p(z)q'(z)}{q(z)^2}$$

Evaluated at a fixed point is

$$\lambda_i = \phi'(z_i) = 1 - \frac{p'(z_i)}{q(z_i)} - 0.$$

Thus we get the linear equation

$$(1-\lambda_i)q(z_i) - p'(z_i) = 0.$$

Note that $\deg(q(z)) = d$ so has $d+1$ coefficients to go along with the $d+1$ fixed points. Finding the coefficients of $q(z)$ when $\lambda_i \neq 1$ is then a question of whether or not the matrix of coefficients of the linear system is invertible. This matrix is a Vandermonde with each row scaled by the nonzero constant $(1-\lambda_i)$. Thus, since the fixed points are distinct, the matrix is invertible and there is a unique solution.

If there is a fixed point at infinity, then the argument is the same except that $\deg(p(z))$ is one less. $\square$

5.1. **Normal Form for Degree 3.** We propose the following normal form for a rational map of degree 3, with distinct fixed points.

Theorem 23.

$$\overline{\phi}(z) = \frac{((-\lambda_1+1)\lambda_0 + (\lambda_1-1))z^3 + (((-\alpha\lambda_1+1)\lambda_0 + (\alpha-1))\lambda_\infty + (((\alpha+1)\lambda_1 - 2)\lambda_0 + (-\lambda_1 + (-\alpha+2))))z^2}{((-\lambda_1+1)\lambda_0 + (\lambda_1-1))\lambda_\infty z^2 + (((\lambda_1-\alpha)\lambda_0 + ((-\alpha-1)\lambda_1 + 2\alpha))\lambda_\infty + ((\alpha-1)\lambda_0 + (\alpha\lambda_1 + (-2\alpha+1))))z}$$
$$\frac{+((\alpha\lambda_1-\alpha)\lambda_0\lambda_\infty + (-\alpha\lambda_1+\alpha)\lambda_0)z)}{+((\alpha\lambda_1-\alpha)\lambda_\infty + (-\alpha\lambda_1+\alpha)))}$$

Where the fixed points are $\{0,1,\infty,\alpha\}$ *with corresponding multipliers* $\{\lambda_0,\lambda_1,\lambda_\infty,\lambda_\alpha\}$ *and*

$$\lambda_\alpha = \frac{1}{\frac{1}{1-\lambda_0} + \frac{1}{1-\lambda_\infty} + \frac{1}{1-\lambda_1} - 1} + 1.$$

Proof. The details are identical to the beginning of the proof of Theorem 20, so we merely state the outline. The explicit calculation was carried out in Pari [10].

We conjugate ϕ so that $\{0,1,\infty\}$ are fixed points with multipliers $\{\lambda_0,\lambda_1,\lambda_\infty\}$. Labeling the last fixed point as α, we have the equation $\phi(\alpha) = \alpha$ allowing us to determine the form of ϕ depending only on the choices of $\{\lambda_0,\lambda_1,\lambda_\infty,\alpha\}$. $\square$

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Ph.D. Program in Mathematics, Graduate Center of CUNY, 365 Fifth Avenue, New York, NY 10016-4309

E-mail address: `bhutz@gc.cuny.edu`

Division of Science and Engineering, Penn State Abington, 1600 Woodland Rd Abington, PA 19001-3918
E-mail address: mlt16@psu.edu